ON THE ASYMPTOTIC CHARACTER OF THE MOTION OF A CONSERVATIVE SYSTEM ACTED ON BY AN APERIODIC PERTURBING FORCE

PMM Vol. 33, №4, 1969, pp. 730-736

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(Received December 16, 1968)

The behavior of the solutions of Eq. (1.1) as $t \to \infty$ in the case of a sufficiently small parameter ε is investigated. It is assumed that the expansion of the function $f(x)$ in powers of x begins with x^2. Familiar methods make it possible to solve the problem in the case where the function $\varphi(t)$ is periodic or almost periodic [1]. The present paper concerns the case where the function $\varphi(t)$ is aperiodic but tends sufficiently rapidly to zero as $t \to +\infty$. It is shown that the solution of the Cauchy problem with zero initial conditions tends asymptotically, as $t \to +\infty$ and for sufficiently small values of the parameter ε, to a certain periodic solution of Eq. (1.10). A series asymptotic in the parameter ε into which the solution can be expanded is constructed.

1. Formulation of the problem. Analysis of the linearized problem. Let us consider the ordinary differential equation

$$x^{\cdot\cdot} + x = f(x) + \varepsilon\varphi(t) \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter and $\varphi(t)$ is a function continuous for $0 \leqslant t < +\infty$ satisfies the following condition for some $\gamma > 0$:

$$\sup \{e^{\gamma t}|\varphi(t)|\} < +\infty, \qquad 0 \leqslant t < +\infty \tag{1.2}$$

The series expansion of the function $f(x)$ in powers of x begins with terms of the order of x^2,

$$f(x) = c_2x^2 + \ldots + c_nx^n + \ldots \qquad (|x| < x_0) \tag{1.3}$$

Equation (1.1) can be interpreted as the equation of motion of a conservative system with one degree of freedom. Let us consider the motions which arise out of the rest state.

In seeking the solution of Eq. (1.1) under zero initial conditions in the form of a series in powers of the small parameter ε, we quickly realize that it is impossible to determine all the coefficients of this series in such a way that they are bounded functions for $0 \leqslant t < +\infty$. We propose a different method of solution.

Let us begin by considering the linear equation

$$x^{\cdot\cdot} + x = F(t) \tag{1.4}$$

The solution of Eq. (1.4) under arbitrary initial conditions is of the form

$$x(t) = \sin t\left[x^{\cdot}(0) + \int_0^t F(\tau)\cos\tau\, d\tau\right] + \cos t\left[x(0) - \int_0^t F(\tau)\sin\tau\, d\tau\right] \tag{1.5}$$

Let the function $F(t)$ satisfy condition (1.2). We introduce the notation

$$\mathrm{S}F = \sin t\int_{+\infty}^t F(\tau)\cos\tau\, d\tau - \cos t\int_{+\infty}^t F(\tau)\sin\tau\, d\tau$$
$$\mathrm{L}_1F = \int_0^{+\infty} F(t)\cos t\, dt, \qquad \mathrm{L}_2F = \int_0^{+\infty} F(t)\sin t\, dt \tag{1.6}$$

This enables us to rewrite expression (1.5) in the form

$$x(t) = \sin t\,[x^{\cdot}(0) + L_1F] + \cos t\,[x(0) - L_2F] + SF \qquad (1.7)$$

Lemma 1.1. The solution of Eq. (1.4) under given initial conditions tends to zero as $t \to +\infty$ if and only if

$$x^{\cdot}(0) + L_1F = 0, \qquad x(0) - L_2F = 0 \qquad (1.8)$$

The proof follows immediately from (1.7) and from the fact that $SF \to 0$ as $t \to +\infty$. If $F = \varepsilon\varphi$, then the solution in the case of zero initial conditions can be written as

$$x(t) = \varepsilon A\cos(t-\alpha) + \varepsilon S\varphi, \qquad L_1\varphi = A\sin\alpha, \qquad L_2\varphi = -A\cos\alpha \qquad (1.9)$$

Formula (1.9) implies that as $t \to +\infty$ the solution of inhomogeneous linear equation (1.4) under zero initial conditions tends to a certain solution of the homogeneous equation. Let us suppose that the solution of nonlinear equation (1.1) tends to a certain periodic solution of the equation

$$x^{\cdot\cdot} + x = f(x) \qquad (1.10)$$

as $t \to +\infty$.

Further on we shall prove the validity of this hypothesis.

2. Reducing the problem to a system of functional equations. The problem of determining the periodic solutions of Eq. (1.10) has been thoroughly investigated. It is sufficient to construct an even solution; any other solution can then be obtained by an arbitrary shift of the independent variable. Following the classical method of Liapunov [1], we set

$$t = \tau\sqrt{a(\mu)}, \qquad a(\mu) = 1 + a_1\mu + \ldots + a_n\mu^n + \ldots \qquad (2.1)$$

and seek the solution in the form of a series in powers of a small parameter,

$$x(\tau) = \mu x_1(\tau) + \ldots + \mu^n x_n(\tau) + \ldots \qquad (2.2)$$

The unknown numbers a_k can be determined from the condition of absence of secular terms in the expressions for x_k. We know that series (2.1) and (2.2) converge for sufficiently small values of the parameter ε, and that series (2.2) converges uniformly. The series obtainable by term-by-term differentiation of (2.2) also converge uniformly. Let us write out the explicit expressions for the first coefficents in expansions (2.1) and (2.2),

$$a_1 = 0, \qquad a_2 = {}^5\!/_6 c_2 + {}^3\!/_4 c_3$$
$$x_1(\tau) = \cos\tau, \qquad x_2(\tau) = c_2\,({}^1\!/_2 - {}^1\!/_6\cos 2\tau) \qquad (2.3)$$

Shifting the independent variable and converting back to the variable t, we can write out the periodic solution of Eq. (1.10) in the form

$$u(t, \mu, b) = \sum_{k=1}^{\infty} x_k[\omega(\mu)(t-b)]\,\mu^k$$
$$\omega(\mu) = [a(\mu)]^{-1/2} = 1 - \mu^2\,({}^5\!/_{12}c_2 + {}^3\!/_8 c_3) + \ldots \qquad (2.4)$$

The values of the function $u(t, \mu, b)$ and its derivative for $t = 0$ are analytic functions of the parameters μ and b,

$$u(0, \mu, b) = \sum_{k=1}^{\infty} x_k(\omega b)\,\mu^k, \qquad u^{\cdot}(0, \mu, b) = \sum_{k=1}^{\infty} \omega x_k^{\cdot}(-\omega b)\,\mu^k \qquad (2.5)$$

Making use of formulas (2.3), we can rewrite (2.5) as

$$u(0, \mu, b) = \mu\cos b + \zeta_1(\mu, b)\mu^2, \qquad u^{\cdot}(0, \mu, b) = \mu\sin b + \zeta_2(\mu, b)\,\mu^2 \qquad (2.6)$$

where $\zeta_1(\mu, b)$ and $\zeta_2(\mu, b)$ are analytic functions of the parameter μ (in the neighborhood of $\mu = 0$) and any finite parameter b. We know that the expansion of the function $u(t, \mu, b)$ in powers of the parameter μ contains secular terms,

$$u(t, \mu, b) = \sum_{k=1}^{\infty} \mu^k u_k(t-b)$$

$$u_1(t) = \cos t, \qquad u_2(t) = c_2(1/2 - 1/6 \cos 2t) \tag{2.7}$$

where $u_k(t-b)$ contains a secular term of the order t^{k-2}.

Lemma 2.1. There exists a number $\mu_0 > 0$ such the the following estimates are valid for $|\mu| < \mu_0$:

$$|u(t, \mu, b)| < B_1|\mu|, \qquad \left|\frac{\partial u(t, \mu, b)}{\partial b}\right| < B_2|\mu|, \qquad \left|\frac{\partial u(t, \mu, b)}{\partial \mu}\right| < B_3\sqrt{1+t^2} \tag{2.8}$$

where the constants B_1, B_2, B_3 do not depend on μ and b.

The proof follows directly from formulas (2.4) and from the fact that the series obtainable by differentiating series (2.4) converges uniformly in t.

Let us assume that μ_0 is so small that $B_1\mu_0 < x_0$, so that, by virtue of (2.8) we also have $|u| < x_0$, where x_0 is defined in expansion (1.3).

Now let us make the following substitution of varaibles in Eq. (1.1):

$$x(t) = u(t, \mu, b) + y(t) \tag{2.9}$$

In order to ensure that the function $x(t)$ satisfies the zero initial conditions we require that

$$y(0) = -u(0, \mu, b), \qquad y^{\cdot}(0) = -u^{\cdot}(0, \mu, b) \tag{2.10}$$

Recalling that $u^{\cdot\cdot} + u = f(u)$, we can rewrite the equation for y in the form

$$y^{\cdot\cdot} + y = f(u+y) - f(u) + \varepsilon\varphi(t) \tag{2.11}$$

The parameters μ and b are as yet arbitrary. Let us assume that they are functions of the patameter ε such that

$$\lim_{t\to+\infty} y(t) = 0 \tag{2.12}$$

By virtue of Lemma 1.1 the solution of Eq. (2.11) under initial conditions (2.10) and condition (2.12) is equivalent to the solution of the system of functional equations

$$y(t) = \varepsilon S\varphi + S[f(u+y) - f(u)] \tag{2.13}$$

$$u(0, \mu, b) - A\varepsilon\cos\alpha + L_2[f(u+y) - f(u)] = 0 \tag{2.14}$$

$$u^{\cdot}(0, \mu, b) - A\varepsilon\sin\alpha - L_1[f(u+y) - f(u)] = 0 \tag{2.15}$$

where the operator S and the functionals L_1 and L_2 are defined by Eqs. (1.6) and the numbers A and α by Eqs. (1.9).

We must determine the three unknown functions $y(t, \varepsilon)$, $\mu(\varepsilon)$ and $b(\varepsilon)$ from the system of three functional equations (2.13)–(2.15). If this is possible, then the hypothesis formulated at the end of the preceding section is valid.

3. Constructing the asymptotic series. Let us attempt to find the formal solution of system (2.13)–(2.15) in the form of series in powers of the small parameter ε,

$$\mu = \sum_{k=1}^{\infty} \varepsilon^k \mu_k, \qquad b = \sum_{k=1}^{\infty} \varepsilon^k b_{k+1}, \qquad y(t) = \sum_{k=1}^{\infty} \varepsilon^k y_k(t) \tag{3.1}$$

From now on we shall denote the k-dimensional vector $(a_1, \ldots a_k)$ by $\mathbf{a}_k$. Substituting expansion (3.1) into (2.7), we obtain

$$u(t, \mu, b) = \sum_{k=1}^{\infty} v_k(t, \mu_k, \mathbf{b}_k)\varepsilon^k, \qquad v_1(t) = \cos t \tag{3.2}$$

where v_k contains a secular term of the order t^{k-2}, $k \geqslant 2$.

Since the function $f(x)$ is analytic, it follows that

$$f(u+y) - f(u) = \sum_{k=1}^{\infty} y^k \frac{f^{(k)}(u)}{k!} \qquad (|y| + |u| < x_0) \tag{3.3}$$

The expansion of y^k in powers of the parameter ε is of the form

$$y^k = \sum_{i=1}^{\infty} P_{ik}(\mathbf{y}_i)\varepsilon^{k+i-1}, \qquad P_{ik}(\mathbf{y}_i) = \sum_{\substack{n_1,\ldots,n_k=1 \\ n_1+\ldots+n_k=i}}^{i-k+1} y_{n_1}, \ldots, y_{n_k} \tag{3.4}$$

The expansion of the function $f'(u)$ can be written as

$$f'(u) = \sum_{j=1}^{\infty} \varepsilon^j f_{j1}(\mathbf{v}_j), \quad f_{j1}(\mathbf{v}_j) = \sum_{\substack{i=1 \\ i+l-2=j}}^{j} \sum_{l=2}^{j+1} l c_l P_{i,\, l-1}(\mathbf{v}_i) \tag{3.5}$$

Similarly,

$$f^{(k)}(u) = \sum_{i=0}^{\infty} \varepsilon^j f_{jk}(\mathbf{v}_j), \quad f_{jk}(\mathbf{v}_j) = \sum_{i=1}^{j+1} \sum_{l=k}^{j+k} l \ldots (l-k+1) c_l P_{i,\, l-k}(\mathbf{v}_i) \tag{3.6}$$

From (3.5) and (3.1) we obtain

$$yf'(u) = \sum_{m=1}^{\infty} \varepsilon^m F_{m,1}(\mathbf{v}_{m-1}, \mathbf{y}_{m-1}), \quad F_{m,1} = \sum_{i=1}^{m-1} \sum_{j=1}^{m-1} f_{j1}(\mathbf{v}_j) y_i \tag{3.7}$$

Expressions (3.6) and (3.4) yield

$$y^k f^{(k)}(u) = \sum_{m=k}^{\infty} \varepsilon^m F_{m,k}(\mathbf{v}_{m-k}, \mathbf{y}_{m-k+1}), \quad F_{m,k} = \sum_{i=1}^{m+1-k} \sum_{j=0}^{m-k} P_{ik}(\mathbf{y}_i) f_{jk}(\mathbf{v}_j) \tag{3.8}$$

Substituting expansions (3.7) and (3.8) into (3.3), we obtain

$$f(u+y) - f(u) = \sum_{n=2}^{\infty} \varepsilon^n \Phi_n(\mathbf{y}_{n-1}, \mathbf{v}_{n-1}) \tag{3.9}$$

$$\Phi_n(\mathbf{y}_{n-1}, \mathbf{v}_{n-1}) = F_{n,1}(\mathbf{v}_{n-1}, \mathbf{y}_{n-1}) + \sum_{k=2}^{n} \sum_{\substack{m=2 \\ m+k-2=n}}^{n} \frac{1}{k!} F_{n,k}(\mathbf{v}_{m-2}, \mathbf{y}_{m-1})$$

Substituting expansions (4.1) into expressions (2.6), we obtain

$$\begin{aligned} u(0, \mu, b) &= \sum_{k=1}^{\infty} [\mu_k \cos b_1 - \mu_1 b_k \sin b_1 + \varphi_k(\boldsymbol{\mu}_{k-1}, \mathbf{b}_{k-1})]\, \varepsilon^k \\ u^{\cdot}(0, \mu, b) &= \sum_{k=1}^{\infty} [\mu_k \sin b_1 + \mu_1 b_k \cos b_1 + \psi_k(\boldsymbol{\mu}_{k-1}, \mathbf{b}_{k-1})]\, \varepsilon^k \end{aligned} \tag{3.10}$$

Substituting expansions (3.9) and (3.10) into Eqs. (2.13)–(2.15) and equating the coefficients of equal powers of the parameter ε, we obtain the recursive system of equations

$$
\begin{aligned}
&y_1(t) = S\varphi, \quad \mu_1 = A, \quad b_1 = \alpha, \quad y_n(t) = S\Phi_n(y_{n-1}, v_{n-1}) \qquad (n = 2, \ldots)\\
&\mu_n \cos\alpha - Ab_n \sin\alpha = -\varphi_n(\mu_{n-1}, b_{n-1}) - L_2\Phi_n(y_{n-1}, v_{n-1})\\
&\mu_n \sin\alpha + Ab_n \cos\alpha = -\psi_n(\mu_{n-1}, b_{n-1}) + L_1\Phi_n(y_{n-1}, v_{n-1})
\end{aligned} \tag{3.11}
$$

If the operator S and the functionals L_1 and L_2 can be applied to the functions $\Phi_n(y_{n-1}, v_{n-1})$ at each step, then all systems (3.11) can be solved successively. This means that formal series (3.1) can, in fact, be constructed.

From now it will be convenient for us to use the terminology of functional analysis. We denote the set of functions continuous for $0 \leqslant t < +\infty$ and satisfying condition (1.2) by B°. This set is a Banach space if the quantity appearing in the left side of inequality (1.2) is regarded as the norm of the function $\varphi(t)$. Similarly, B^n is the Banach space of functions continuous for $0 \leqslant t < +\infty$ with the norm

$$
\|\varphi\|_{B^n} = \sup\{e^{\gamma t}|\varphi(t)|(1+t^2)^{-1/2n}\} \qquad (0 \leqslant t < +\infty) \tag{3.12}
$$

Lemma 3.1. The operator S acting from the space B^n into B^n for any n and is bounded. The functionals L_1 and L_2 are also bounded in B^n.

The proof of this Lemma is elementary. It requires the use of explicit expressions (1.6) for the operator S and of the functionals L_1 and L_2.

Theorem 3.1. If $\varphi \in B^\circ$, then all of the equations of (3.11) can be solved successively; moreover, $y_1 \in B^\circ$, $y_2 \in B^\circ$, $y_{n+3} \in B^n$ for $n = 0, 1 \ldots$

Proof. If $\varphi \in B^\circ$, then by Lemma 3.1 we have $y_1 = S\varphi \in B^\circ$. Further, $y_2 = S\Phi_2(y_1, v_1)$, where $\Phi_2(y, v_1)$ is a poynomial which does not contain the zeroth power of y_1. The function $v_1(t)$ is bounded, so that $\Phi_2(y_1, v_1) \in B^\circ$, and we infer by Lemma 3.1 that $y_2 = S\Phi_2(y_1 v_1) \in B^\circ$. It can be shown in similar fashion that $y_3 \in B^\circ$. We shall prove the theorem by induction. Let $y_3 \in B^\circ, \ldots, y_{n+3} \in B^n$. Let us prove that this implies that $y_{n+4} \in B^{n+1}$. Since $v_k(t)$ contains a secular term of the order t^{k-2} for $k \geqslant 2$, it follows by formula (3.4) that $P_{ik}(v_i)$ contains a secular term of the order t^{i-2} for $i \geqslant 2$. From (3.5) and (3.6) we infer that $f_{jk}(v_j)$ contains a secular term of the order $t^{j-2}, j \geqslant 2$. We now infer from (3.7) and (3.8) that $F_{n+4,k}(v_{n+4-k}, y_{n+5-k}) \in B^{n+1}$. From (3.9) we infer that $\Phi_{n+4}(y_{n+3}, v_{n+1}) \in B^{n+1}$, and since $y_{n+4} = S\Phi_{n+4}$, it follows that $y_{n+4} \in B^{n+1}$, QED.

This enables us to construct series (3.1). We shall show that these series are asymptotic on the parameter ε.

The expressions involved in the prctical computations of the coefficients of series (3.1) are extremely cumbesome. Let us write out the formulas for the quantities $y_2(t)$, μ_2 and b_2,

$$
\begin{aligned}
y_2(t) &= c_2 S[(S\varphi)^2 + 2A\cos(t-\alpha)S\varphi]\\
\mu_2 &= {}^1/_6 A^2 c_2[2\sin\alpha\sin 2\alpha - 3\cos\alpha + \cos\alpha\cos 2\alpha] -\\
&\quad - c_2\int_0^\infty \sin(t-\alpha)[(S\varphi)^2 + 2A\cos(t-\alpha)S\varphi]\,dt\\
b_2 &= {}^1/_6 Ac_2[2\sin 2\alpha\cos\alpha + 3\sin\alpha - \cos 2\alpha\sin\alpha] +\\
&\quad + c_2\int_0^\infty \cos(t-\alpha)[(S\varphi)^2 + 2A\cos(t-\alpha)S\varphi]\,dt
\end{aligned} \tag{3.13}
$$

4. Proof of the asymptotic character of series (3.1). Let us set

$$B = \sum_{k=0}^{N-1} \varepsilon^k b_{k+1}, \qquad v = \sum_{k=1}^{N} \varepsilon^k \mu_k, \qquad Y(t) = \sum_{k=1}^{N} \varepsilon^k y_k(t)$$

$$U(t, \varepsilon) = u(t, \nu, B), \qquad u = U + u^* \varepsilon^N$$

and seek the solution of system (2.13)–(2.15) in the form

$$y = Y + y^* \varepsilon^N, \qquad b = B + b^* \varepsilon^{N-1}, \qquad \mu = \nu + \mu^* \varepsilon^N \tag{4.1}$$

Lemma 4.1. There exist positive numbers ε_0, η_0 b_0, μ_0 such that for

$$|\varepsilon| < \varepsilon_0, \quad |\mu^*| < \mu_0, \quad |b^*| < b_0, \quad \| y^* \|_{B^k} < \eta_0(k) \tag{4.2}$$

we have the inequalities

$$|\nu| < A_1\varepsilon, \quad \|Y(t)\|_{B^{N-3}} \leqslant A_2\varepsilon, \qquad |\mu| < A_3\varepsilon$$

$$\|y\|_{B^k} \leqslant A_4(k)\varepsilon, \qquad |u(t, \mu, b)| < A_5\varepsilon, \qquad |U(t, \varepsilon)| < A_6\varepsilon \tag{4.3}$$

$$|u^*| \leqslant A_7(|\mu^*| + |b^*|)\sqrt{1+t^2} \qquad (k \geqslant N-3)$$

where the constants A_i do not depend on ε, μ^* and b^*.

Proof. All of the above inequalities except the last are readily derivable from formulas (4.1). Let us prove the last inequality. Applying the finite increment formula, we obtain

$$u(t, \mu, b) - u(t, \nu, B) = (\mu - \nu)\, u'_\mu\, [t, \nu + \theta(\mu - \nu), B + \theta(b - B)] + (b - B)$$
$$u_b'\, [t, \nu + \theta(\mu - \nu), B + \theta(b - B)] \qquad (0 < \theta < 1)$$

Making use of Eqs. (4.1) and inequalities (2.8), we obtain

$$|u^*| \leqslant |\mu^*| B_3 \sqrt{1+t^2} + |b^*| \frac{|\nu| + \varepsilon^N |\mu^*|}{\varepsilon} \tag{4.4}$$

Expressions (4.3) and (4.4) imply that the last inequality of (4.3) is valid for sufficiently small ε. Let us consider the function

$$\varphi(u, y) = f(u + y) - f(u) \tag{4.5}$$

Lemma 4.2. We can choose the numbers $\varepsilon_0, \mu_0, b_0$ and η_0 in such a way that fulfilment of inequalities (4.2) for the function $\varphi(u, y)$ implies the validity of the estimates

$$\|\varphi(u, y) - \varphi(U, Y)\|_{B^{N-2}} \leqslant A_8 \varepsilon^{N+1}$$

$$\|\varphi(u, Y + \varepsilon^N y_1^*) - \varphi(u, Y + \varepsilon^N y_2^*)\|_{B^{N-2}} \leqslant A_9 \| y_2^* - y_1^* \|_{B^{N-2}} \varepsilon^{N+1} \tag{4.6}$$

where the constants A_8 and A_9 do not depend on ε, μ^* and b^*.

Proof. Expansion (1.3) implies that the following inequalities are fulfilled for the function $f(x)$ for $|x| < x_0$:

$$|f'(x)| < c_1 |x|, \qquad |f(x_1) - f(x_2)| \leqslant c_2 |x_2 - x_1| \tag{4.7}$$

Let us choose the numbers ε_0, b_0, η_0 and μ_0 in such a way as to ensure fulfilment of inequalities (4.3) and the inequality $|u| + |y| < x_0$. By virtue of (4.7) this means that

$$\left|\frac{\partial \varphi(u, y)}{\partial u}\right| = |f'(u+y) - f'(u)| \leqslant c_2 |y|, \quad \left|\frac{\partial \varphi(u, y)}{\partial u}\right| = |f'(u+y)| \leqslant C_1(|u| + |y|) \tag{4.8}$$

Applying the finite increment formula, we obtain

$$\varphi(u, y) - \varphi(U, Y) = (u - U)\frac{\partial \varphi}{\partial u}[U + \theta(u - U), \quad Y + \theta(y - Y)] +$$
$$+ (y - Y)\frac{\partial \varphi}{\partial y}[U + \theta(u - U), Y + \theta(y - Y)] \qquad (0 < \theta < 1)$$

Making use of inequalities (4.8), we find that

$$\varphi(u, y) - \varphi(U, Y)| < \varepsilon^N |u^*| c_2 (|Y| + |y|) + \varepsilon^N |y^*| (|u| + |U| + |y| + |Y|)$$

Applying inequality (4.3) for u^*, we obtain

$$\|\varphi(u, y) - \varphi(U, Y)\|_{B^{N-2}} \leqslant A_7 C_2 \varepsilon^N (|\mu^*| + |b^*|)(\|Y\|_{B^{N-3}} + \|y\|_{B^{N-3}}) +$$
$$+ \varepsilon^N \|y\|_{B^{N-3}} \sup \{(|u| + |U| + |y| + |Y|)(1 + t^2)^{-1/2}\}, 0 \leqslant t < +\infty \qquad (4.9)$$

But by virtue of Lemma 4.1 for sufficiently small ε, μ^*, b^* and η_0 we have

$$\|Y\|_{B^{N-3}} \leqslant A_2\varepsilon, \|y\|_{B^{N-3}} \leqslant A_4\varepsilon, |u| < A_5\varepsilon, |U| < A_6\varepsilon, |y| \leqslant \|y\|_{B^{N-3}} e^{-\gamma t}(1 + t^2)^{1/2(N-3)} < A_{10}\varepsilon$$
$$|Y| \leqslant \|Y\|_{B^{N-3}} e^{-\gamma t}(1 + t^2)^{1/2(N-3)} \leqslant A_{11}\varepsilon$$

Substituting these estimates into (4.9), we obtain the first inequality of (4.6). The second inequality can be obtained in similar fashion. It is clear that the first N coefficients of the expansions of the functions $\varphi(u, y)$ in powers of ε coincide with the first N coefficients of the expansion of the function $\varphi(U, Y)$. By virtue of (3.9) we have

$$\varphi(U, Y) = \sum_{n=2}^{N} \varepsilon^n \Phi_n (\mathbf{y}_{n-1}, \mathbf{v}_{n-1}) + \varepsilon^{N+1} \chi(t, \varepsilon) \qquad (4.10)$$

where $\chi(t, \varepsilon)$ is some known continuous function. Since $Y \in B^{N-3}$, and since U is a bounded function, it follows that $\varphi(U, Y) \in B^{N-3}$. In precisely similar fashion we obta

$$\sum_{n=2}^{N} \varepsilon^n \Phi_n (\mathbf{y}_{n-1}, \mathbf{v}_{n-1}) \in B^{N-3}$$

Hence, $\chi(t, \varepsilon) \in B^{N-3}$ and (4.10) yields the following equation for $\varphi(u, y)$:

$$\varphi(u, y) = \varphi(u, y) - \varphi(U, Y) + \sum_{n=2}^{N} \varepsilon^n \Phi_n (\mathbf{y}_{n-1}, v_{n-1}) + \varepsilon^{N+1} \chi(t, \varepsilon) \qquad (4.11)$$

Substituting expression (4.11) into Eq. (2.13) and making use of Eqs. (3.11) for $n = 2, \ldots N$, we obtain an equation for y^*,

$$y^* = \varepsilon S \Phi^* (y^*, \mu^*, b^*, \varepsilon), \quad \Phi^* = \varepsilon^{-(N+1)} [\varphi(u, y) - \varphi(U, Y)] + \chi(t, \varepsilon) \quad (4.12)$$

Theorem 4.1. There exist numbers $\varepsilon_0, \mu_0, b_0$ and η_0 such that fulfilment of inequalities (4.2) implies that Eq. (4.12) has the unique solution $y^* = y^*(t, \varepsilon, \mu^*, \cancel{b^*)},$ where y^* is a differentiable function of μ^*, b^*, and where the following estimates are valid:

$$\|y^*\|_{B^{N-2}} \leqslant C_1\varepsilon, \quad \left\|\frac{\partial y^*}{\partial b^*}\right\|_{B^{N-2}} \leqslant C_2\varepsilon, \quad \left\|\frac{\partial y^*}{\partial \mu^*}\right\|_{B^{N-1}} \leqslant C_3\varepsilon \qquad (4.13)$$

Theorem 4.1 is a simple consequence of Lemma 4.2 and of the principle of compressed mappings [2].

Now let us derive the equations for determining μ^* and b^*. We note that the first N coefficients of the expansions of the functions $u(0, \mu, b)$ and $u(0, \nu, B)$ in powers of ε coincide. The first N coefficients of the expansions of the functions $u^{\cdot}(0, \mu, b)$ and $u^{\cdot}(0, \nu, B)$ also coincide. Recalling Eqs. (3.10), we obtain

$$u(0, \nu, B) = \sum_{k=1}^{\infty} [\mu_k \cos\alpha - Ab_k \sin\alpha + \varphi_k(\mu_{k-1}, \mathbf{b}_{k-1})] \varepsilon^k + \varepsilon^{N+1} \varphi^*(\varepsilon) \qquad (4.14)$$
$$u^{\cdot}(0, \nu, B) = \sum_{k=1}^{\infty} [\mu_k \sin\alpha + AB_k \cos\alpha + \psi_k(\mu_{k-1}, \mathbf{b}_{k-1})] \varepsilon^k + \varepsilon^{N+1} \psi^*(\varepsilon)$$

where $\varphi^*(\varepsilon)$ and $\psi^*(\varepsilon)$ are some continuous functions of the parameter ε. Substituting expansion (4.14) into Eqs. (2.14) and (2.15), we obtain the following equations for determining μ^* and b^*:

$$u^*(0, \mu^*, b^*) - \varepsilon[\varphi^*(\varepsilon) - L_2\Phi^*(y^*, \mu^*, b^*, \varepsilon)] = 0$$
$$u^{\cdot *}(0, \mu^*, b^*) - \varepsilon[\psi^*(\varepsilon) + L_1\Phi^*(y^*, \mu^*, b^*, \varepsilon)] = 0 \qquad (4.15)$$
$$\text{где } u^* = \varepsilon^{-N}[u(0, \mu, b) - u(0, \nu, B)]$$

Substituting the function $y^*(t, \mu^*, b^*, \varepsilon)$ determined from Eq. (4.12) into Eq. (4.15), we obtain a system of two equations for determining μ^* and b^*. Lemma 4.1 implies that $|u^*(0, \mu^*, b^*)| \leqslant c(|\mu^*| + |b^*|)$. Hence, $u^*(0, 0, 0) = 0$. It is easy to show that $u^{\cdot *}(0, 0, 0) = 0$. Hence, Eqs. (4.15) have the solution $\mu^* = b^* = 0$ for $\varepsilon = 0$. Let us show that the Jacobian of this system is different from zero for $\varepsilon = \mu^* = b^* = 0$. In fact,

$$\frac{\partial u^*(0, \mu^*, b^*)}{\partial \mu^*} = \frac{1}{\varepsilon^N}\frac{\partial u(0, \mu, b)}{\partial \mu}\frac{\partial \mu}{\partial \mu^*} = \frac{\partial u(0, \mu, b)}{\partial \mu}$$

Recalling expression (2.6) for $u(0, \mu, b)$, we obtain

$$\left(\frac{\partial u^*}{\partial \mu^*}\right)_{\mu^*=b^*=\varepsilon=0} \equiv \left(\frac{\partial u^*}{\partial \mu^*}\right)_0 = \cos\alpha \qquad (4.16)$$

Similarly, we can show that

$$\left(\frac{\partial u^*}{\partial b^*}\right)_0 = -A\sin\alpha, \quad \left(\frac{\partial u^{\cdot *}}{\partial \mu^*}\right)_0 = \sin\alpha, \quad \left(\frac{\partial u^{\cdot *}}{\partial b^*}\right)_0 = A\cos\alpha \qquad (4.17)$$

Expressions (4.16) and (4.17) imply that for $\varepsilon^* = \mu^* = b^* = 0$ the Jacobian of system (4.15) is equal to the number $A > 0$. By the implicit function theorem, system (4.15) has the solution $\mu^*(\varepsilon)$, $b^*(\varepsilon)$ in a sufficiently small neighborhood of the point $\varepsilon = \mu^* = b^* = 0$; moreover, $\mu^*(\varepsilon)$ and $b^*(\varepsilon)$ tend to zero as $\varepsilon \to 0$.

We have thus proved the following theorem.

Theorem 4.2. The solution of system (2.13)–(2.15) can always be found in the form (4.1), where $b^*(\varepsilon)$, $\mu^*(\varepsilon)$ and $\|y^*\|_{B^{N-2}}$ tend to zero as $\varepsilon \to 0$. Hence, series (3.1) are asymptotic.

The proposed method is also suitable for investigating conservative systems with many degrees of freedom.

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Translated by A. Y.